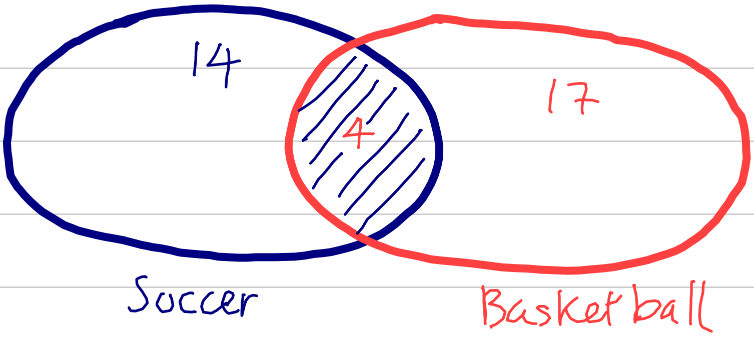
Chapter 7. Principle of Inclusion-Exclusion

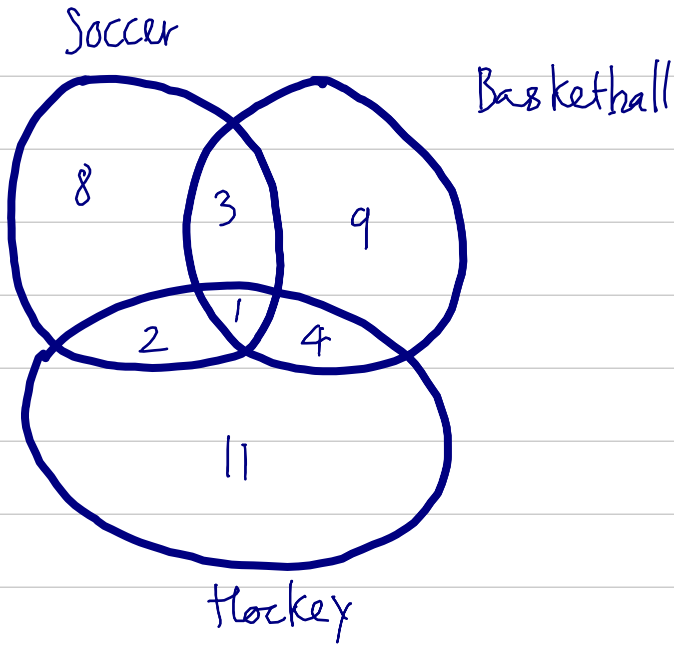
Example 7.1 There are 14 students in a high school class who play soccer, and there are 17 students who play basketball. Four students play both games. How many students play at least one of the two games?



Answer: is $14 + 17 - 4 = 27$,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example 7.2. In a high school class, there are 14 students who play soccer, 17 students who play basketball, and 18 students who play hockey. Four students play both soccer an basketball, three play both soccer an hockey, and five play both basketball and hockey. There is one student who plays all three games. How many students play at least one of these games?



Answer: $14+17+18-4-3-5+1=38$

$$|A \cup B \cup C| = |A|+|B|+|C| - |A\cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

Theorem (Sieve Formula)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, i_2, \ldots, i_j} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}|$$

where $\{i_1, i_2, \ldots, i_j\}$ ranges over all j-subsets of $[n]$.

Proof. We show that each element of $A_1 \cup \ldots \cup A_n$ is counted exactly once on the RHS.

Let $x \in A_1 \cup \ldots \cup A_n$. Assume that $S \subseteq [n]$ is the set of indices i s.t. $x \in A_i$. Denote by $s = |S|$. ≥ 1.

We have $x \in A_{i_1} \cap \ldots \cap A_{i_j}$ iff $(i_1, \ldots, i_j) \subseteq S$.

There are s set $A_i \ni x$. $\to$ contributes s to the counting of x.

There are $\binom{s}{2}$ pair (i_1, i_2) s.t. $x \in A_{i_1} \cap A_{i_2}$

$\Rightarrow$ contribute $-\binom{s}{2}$ to the counting of x on the RHS.

...

There are $\binom{s}{k}$ k-tuples $(i_1, \dots, i_k)$ s.t. $x \in A_{i_1} \cap \dots \cap A_{i_k}$

$\Rightarrow$ contributes $(-1)^{k-1}\binom{s}{k}$ to the counting....

Therefore, the RHS counts x

$$s - \binom{s}{2} + \binom{s}{3} - \binom{s}{4} + \dots + (-1)^{s-1}\binom{s}{s} = 1$$

times, for any x. $\square$

Example 7.3. A party has n guests. When the guests arrived, they left their hats in the same coatroom. After the party ended, there was an electrical failure, so guests took a hat from the coatroom at random. When the guests were back on the street, they were amused to find out that none of them got his hat back. In how many different ways could that happen?

Comments: + For $x \in [n]$ s.t. $p(x) = x$, we say that x is a **fixed point** of p.

+ Our problem is to count the permutations with no fix point. Such a permutation is called a **dearrangement**. The number of all dearrangements of $[n]$ is denoted by $D(n)$.

Solution: Denote by A_i the set of all n-permutations whose a fix point is i ($i = 1, 2 \ldots, n$).

Thus, the number of all permutations with at least one fix point is $|A_1 \cup \ldots \cup A_n|$. Then

$$D(n) = n! - |A_1 \cup \ldots \cup A_n|.$$

By the Sieve Formula, we have

$$|A_1 \cup \ldots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, \ldots, i_j} |A_{i_1} \cap \ldots \cap A_{i_j}|$$

we have $A_{i_1} \cap \ldots \cap A_{i_j}$ is the set of permutations with j fixed points: $i_1, \ldots, i_j$. This set is in bijection with the set of $(n-j)$-permutations.

$\Rightarrow$

$$|A_{i_1} \cap \ldots \cap A_{i_j}| = (n-j)! \quad \text{for any}$$

j-tuples $(i_1, \ldots, i_j)$

$$\Rightarrow \sum_{i_1, \ldots, i_j} |A_{i_1} \cap \ldots \cap A_{i_j}| = \binom{n}{j} (n-j)!$$

$$= \frac{n!}{j!}.$$

$\Rightarrow$

$$|A_1 \cup \ldots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \frac{n!}{j!}$$

$$\Rightarrow D(n) = \sum_{j=0}^{n} (-1)^{j} \frac{n!}{j!}. \qquad \square$$

We have $\frac{D(n)}{n!} = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \to \frac{1}{e}$ as $n \to \infty$.

Example 7.4. For all positive integers n and k

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^k \binom{k}{i} (k-i)^n$$
$$= \sum_{i=0}^{k} (-1)^i \frac{1}{i!(k-i)!} (k-i)^n.$$

Proof:

Recall: The number of all surjections from $[n]$ onto $[k]$ is $k!\,S(n,k)$.

We will show that $k!\,S(n,k) = \sum_{i=0}^{k} (-1)^k \binom{k}{i} (k-i)^n$.

OR, show that the RHS counts all surjections from $[n]$ onto $[k]$.

Let A_i be the set of functions from $[n]$ to $[k]$, such that the image does not contain i,

$\Rightarrow$ the set of all functions that are not onto is $A_1 \cup \dots \cup A_n$.

We have $|A_1 \cup \dots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum |A_{i_1} \cap \dots \cap A_{i_j}|$

$$= \sum_{j=1}^{n} (-1)^{j-1} \binom{k}{j} (k-j)^n$$

$\Rightarrow$ The number of surjections from $[n]$ onto $[k]$ is

$$k^n - |A_1 \cup \dots \cup A_n| = k^n - \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} (k-i)^n$$
$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n. \quad \square.$$

Example 7.5. Let f and g be functions that are defined on the subsets of $[n]$, and whose range is the set of real numbers. Let us assume that f and g are connected by

$$g(S) = \sum_{T \subseteq S} f(T).$$

Then

$$f(S) = \sum_{T \subseteq S} g(T) (-1)^{|S-T|}. \quad (\star)$$

Proof. If we express $g(T)$ by value of f on the RHS of $(\star)$, we need to show that

$$f(S) = \sum_{U \subseteq T \subseteq S} f(U) (-1)^{|S-T|}. \quad (\star\star)$$

We see that each $f(U)$ appears exactly once for each set T satisfying $U \subseteq T \subseteq S$. Each such appearance of $f(U)$ will be counted by $(-1)^{|S-T|}$. The number of such sets T for which $|S-T| = i$ is equal to $\binom{|S-U|}{i}$, since T is determined by the elements of S that are not in T, and T contains U.

There for $f(U)$ appear on RHS of $(\star\star)$ exactly

$$\sum_{i=0}^{|S-U|} (-1)^i \binom{|S-U|}{i} = (1-1)^{|S-U|} \quad \text{times}.$$

This number is always zero, except for $U = S$.

$\Rightarrow$ $(\star\star)$ holds. $\square$.